TABLE 8.9 PARAMETERS FOR ELEMENT EQUATIONS

Element (e)	Coordinates							
	x_i	x_j	x_k	x_l	y_i	y_j	y_k	y_l
4	1/2	1	$\sqrt{3}/2$	$\sqrt{15}/4$	1/2	0	1/2	1/4
6	0	1/2	1/2	1/4	1	1/2	$\sqrt{3}/2$	$\sqrt{15}/4$
7	1/2	$\sqrt{3}/2$	1/2	$\sqrt{7}/4$	1/2	1/2	$\sqrt{3}/2$	3/4

Element (e)	Parameters					
	b_i $y_j - y_k$	b_j $y_k - y_i$	b_k $y_i - y_j$	c_i $x_k - x_j$	c_j $x_i - x_k$	c_k $x_j - x_i$
4	$-1/2$	0	1/2	$(\sqrt{3}-2)/2$	$(1-\sqrt{3})/2$	1/2
6	$(1-\sqrt{3})/2$	$(\sqrt{3}-2)/2$	1/2	0	$-1/2$	1/2
7	$(1-\sqrt{3})/2$	$(\sqrt{3}-1)/2$	0	$(1-\sqrt{3})/2$	0	$(\sqrt{3}-1)/2$

There are six elements containing node 5.
Each element will contribute to the nodal equation determining u_5. The contribution from the elements is as follows:

Element (2)	$u_5 = 0.06875$
Element (3)	$0.5u_5 = 0.075$
Element (5)	$0.5u_5 = 0.075$
Element (4)	$1.5342196u_5 = 0.0426976$
Element (6)	$0.6484375u_5 = 0.1090707$
Element (7)	$0.6484375u_5 = 0.1900744$

By summing the above equations we get

$$4.8310946u_5 = 0.5605927$$

which gives

$$u_5 = 0.1160384$$

Example 8.10 Use finite element method to solve the boundary value problem

$$\nabla^2 u = -1, \quad |x| \leqslant 1, \quad |y| \leqslant 1$$

$$\frac{\partial u}{\partial n} + u = 0, \quad |x| = 1, \quad |y| = 1$$

with $h = \frac{1}{2}$.

On account of symmetry, we need only consider one eighth of the square. We use the triangular net as shown in Figure 8.15(a). The nodal parameters are given in Table 8.7.

TABLE 8.10 NODAL PARAMETERS FOR ELEMENT EQUATIONS

Element (e)	Nodes i	j	k
1	1	2	4
2	4	2	5
3	2	3	5
4	5	3	6
5	4	5	7
6	7	5	8
7	5	6	8
8	8	6	9

Element (e)	Coordinates x_i	x_j	x_k	y_i	y_j	y_k
1	0	1/2	0	0	0	1/2
2	0	1/2	1/2	1/2	0	1/2
3	1/2	1	1/2	0	0	1/2
4	1/2	1	1	1/2	0	1/2
5	0	1/2	0	1/2	1/2	1
6	0	1/2	1/2	1	1/2	1
7	1/2	1	1/2	1/2	1/2	1
8	1/2	1	1	1	1/2	1

Element (e)	Parameters b_i y_j-y_k	b_j y_k-y_i	b_k y_i-y_j	c_i x_k-x_j	c_j x_i-x_k	c_k x_j-x_i
1	−1/2	1/2	0	−1/2	0	1/2
2	−1/2	0	1/2	0	−1/2	1/2
3	−1/2	1/2	0	−1/2	0	1/2
4	−1/2	0	1/2	0	−1/2	1/2
5	−1/2	1/2	0	−1/2	0	1/2
6	−1/2	0	1/2	0	−1/2	1/2
7	−1/2	1/2	0	−1/2	0	1/2
8	−1/2	0	1/2	0	−1/2	1/2

$$\mathbf{A}^{(6)}\boldsymbol{\phi}^{(6)} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_7 \\ u_5 \\ u_8 \end{bmatrix}, \quad \mathbf{b}^{(6)} = \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A}^{(7)}\boldsymbol{\phi}^{(7)} = \tfrac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_5 \\ u_6 \\ u_8 \end{bmatrix}, \quad \mathbf{b}^{(7)} = \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A}^{(8)}\boldsymbol{\phi}^{(8)} = \tfrac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_8 \\ u_6 \\ u_9 \end{bmatrix}, \quad \mathbf{b}^{(8)} = \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The equation (8.227) becomes

$$\begin{bmatrix} 24 & -12 & 0 & -12 & & & & & \\ -12 & 48 & -12 & 0 & -24 & & & & \\ 0 & -12 & 24 & 0 & 0 & -12 & & & \\ -12 & 0 & 0 & 48 & -24 & 0 & -12 & & \\ & -24 & 0 & -24 & 96 & -24 & 0 & -24 & \\ & & -12 & 0 & -24 & 48 & 0 & 0 & -12 \\ & & & -12 & 0 & 0 & 24 & -12 & 0 \\ & & & & -24 & 0 & -12 & 48 & -12 \\ & & & & & -12 & 0 & -12 & 24 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 6 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

The essential boundary conditions are

$$u_3 = u_6 = u_9 = u_8 = u_7 = 1$$

On incorporating $u_3 = 1$, we get

$$\begin{bmatrix} 24 & -12 & 0 & -12 & & & & & \\ -12 & 48 & 0 & 0 & -24 & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & & \\ -12 & 0 & 0 & 48 & -24 & 0 & -12 & & \\ & -24 & 0 & -24 & 96 & -24 & 0 & -24 & \\ & & 0 & 0 & -24 & 48 & 0 & 0 & -12 \\ & & & -12 & 0 & 0 & 24 & -12 & 0 \\ & & & & -24 & 0 & -12 & 48 & -12 \\ & & & & & -12 & 0 & -12 & 24 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 6 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix} - 1 \times \begin{bmatrix} 0 \\ -12 \\ 0 \\ 0 \\ 0 \\ -12 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \\ 1 \\ 3 \\ 6 \\ 15 \\ 2 \\ 3 \\ 1 \end{bmatrix}
$$

In a similar manner we introduce the other boundary conditions and obtain

$$
\begin{bmatrix} 24 & -12 & 0 & -12 & 0 & 0 & 0 & 0 & 0 \\ -12 & 48 & 0 & 0 & -24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 0 & 0 & 48 & -24 & 0 & 0 & 0 & 0 \\ 0 & -24 & 0 & -24 & 96 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \\ 1 \\ 15 \\ 54 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

The equations to determine the unknown can be written as

$$
\begin{bmatrix} 24 & -12 & -12 & 0 \\ -12 & 48 & 0 & -24 \\ -12 & 0 & 48 & -24 \\ 0 & -24 & -24 & 96 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \\ 15 \\ 54 \end{bmatrix}
$$

The solution values are given by

$$u_1 = 1.25, \quad u_2 = 1.21, \quad u_4 = 1.21 \quad u_5 = 1.17$$

8.8 NONLINEAR DIFFERENTIAL EQUATIONS

The finite element methods outlined in Section 8.5 may easily be applied to the nonlinear differential equations. The mathematical difficulties are generally overcome by two approaches. One is to linearize initially the original nonlinear differential equations (see Section 5.11) and then solve the problem by a process of successive linear finite element methods. The other approach is to formulate directly nonlinear finite element equations and then numerically solve the resulting system of nonlinear algebraic equations by the Newton-Raphson method (see Section 4.3). We will now apply these two approaches to a simple nonlinear boundary value problem.

Example 8.12 Use the finite element method to solve the boundary value problem

$$u'' = \frac{3}{2}u^2$$

$$u(0) = 4,\ u(1) = 1$$

with $h = \frac{1}{3}$.

The exact solution of the boundary value problem is

$$u(x) = \frac{4}{(1+x)^2}$$

Applying the Newton linearization process to the boundary value problem, we obtain

$$u''^{(p+1)} = 3u^{(p)}u^{(p+1)} - \frac{3}{2}(u^{(p)})^2$$

where the superscript (p) represents the pth iterate. We now determine $(p+1)$th iterates from the pth iterates. The variation formulation may be written as

$$J = \int_0^1 [(u'^{(p+1)})^2 + 3u^{(p)}(u^{(p+1)})^2 - 3(u^{(p)})^2 u^{(p+1)}]\, dx$$

$$= \text{minimum}$$

The piecewise linear approximate solution over the element (e) for $(p+1)$th iterate is of the form

$$(u^{(p+1)})^{(e)} = N_j u_j^{(p+1)} + N_k u_k^{(p+1)}$$

Differentiating the element functional $J^{(e)}$ with respect to $u_j^{(p+1)}$ and $u_k^{(p+1)}$, we get the element equation

$$\begin{bmatrix} 1 + \frac{h^2}{4}(3u_j^{(p)} + u_k^{(p)}) & -1 + \frac{h^2}{4}(u_j^{(p)} + u_k^{(p)}) \\ -1 + \frac{h^2}{4}(u_j^{(p)} + u_k^{(p)}) & 1 + \frac{h^2}{4}(u_j^{(p)} + 3u_k^{(p)}) \end{bmatrix} \begin{bmatrix} u_j^{(p+1)} \\ u_k^{(p+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{h^2}{8}(3(u_j^{(p)})^2 + 2u_j^{(p)}u_k^{(p)} + (u_k^{(p)})^2) \\ \frac{h^2}{8}((u_j^{(p)})^2 + 2u_j^{(p)}u_k^{(p)} + 3(u_k^{(p)})^2) \end{bmatrix}$$

We assume that the numerical solution of (8.240) has been obtained upto and including at $t = t_i$ and the solution value u_{i+1} is the unknown to be determined. The Galerkin equation (8.16) over the element (e) becomes

$$\int_{t_i}^{t_{i+1}} \frac{(t-t_i)}{h}\left[\frac{du^{(e)}}{dt} - fu^{(e)} - g\right] dt = 0 \tag{8.242}$$

Substituting (8.241) into (8.242) and assuming that the functions $f(t)$ and $g(t)$ are constants in each element, we obtain

$$u_{i+1} = u_i + \frac{h}{3} f^{(e)}(u_i + 2u_{i+1}) + hg^{(e)}$$

or

$$u_{i+1} = \frac{\left(1 + \frac{1}{3}hf^{(e)}\right)}{\left(1 - \frac{2}{3}hf^{(e)}\right)} u_i + \frac{hg^{(e)}}{\left(1 - \frac{2}{3}hf^{(e)}\right)} \tag{8.243}$$

where $f^{(e)}$ and $g^{(e)}$ represent the function values of f and g respectively in each element (e).

Quadratic Lagrange polynomial

We take the approximate function $u^{(e)}$ of the form

$$u^{(e)} = \frac{1}{2h^2}(t-t_i)(t-t_{i+1})u_{i-1} - \frac{1}{h^2}(t-t_{i-1})(t-t_{i+1})u_i$$

$$+ \frac{1}{2h^2}(t-t_i)(t-t_{i-1})u_{i+1} \tag{8.244}$$

where $t_{i+1} - t_{i-1} = 2h$ is the length of the element (e). Here, the Galerkin equation (8.242) becomes

$$\int_{t_{i-1}}^{t_{i+1}} \frac{1}{2h^2}(t-t_i)(t-t_{i-1})\left[\frac{du^{(e)}}{dt} - fu^{(e)} - g\right] dt = 0 \tag{8.245}$$

Substituting (8.244) into (8.245) and simplifying we have

$$u_{i+1} - \frac{4}{3}u_i + \frac{1}{3}u_{i-1} = \frac{h}{15} f^{(e)}(8u_{i+1} + 4u_i - 2u_{i-1}) + \frac{2}{3}hg^{(e)} \tag{8.246}$$

Hermite cubic polynomial

The equation (8.55) becomes

$$u^{(e)} = L_i^2(3 - 2L_i)u_i + hL_i^2 L_{i+1} u_i' + L_{i+1}^2(3 - 2L_{i+1})u_{i+1} - hL_{i+1}^2 L_i u_{i+1}' \tag{8.247}$$

where L_i and L_{i+1} are the local coordinates associated with the element (e) and $L_{i+1} = (t - t_i)/h$.

Keeping in view that the derivative value u'_{i+1} may be determined using (8.240) if the function value u_{i+1} is known, the Galerkin equation for the unknown solution value u_{i+1} becomes

$$\int_{t_i}^{t_{i+1}} L_{i+1}^2(3-2L_{i+1})\left[\frac{du^{(e)}}{dt}-fu^{(e)}-g\right]dt=0$$

which after integration (using (8.54)) is given by

$$u_{i+1}=u_i-\frac{h}{5}(u'_{i+1}-u'_i)$$
$$+hf^{(e)}\left[\frac{9}{35}u_i+\frac{26}{35}u_{i+1}+\frac{13}{210}h\,u'_i-\frac{22}{210}h\,u'_{i+1}\right]+hg^{(e)} \qquad (8.248)$$

Hermite quintic polynomial

We use the approximate function in the form

$$u^{(e)}=(10L_i^3-15L_i^4+6L_i^5)u_i+h(4L_i^3-7L_i^4+3L_i^5)\,u'_i+\frac{1}{2}\,h^2L_i^3(1-L_i)^2u''_i$$
$$+(10L_{i+1}^3-15L_{i+1}^4+6L_{i+1}^5)u_{i+1}-h(4L_{i+1}^3-7L_{i+1}^4+3L_{i+1}^5)\,u'_{i+1}$$
$$+\frac{1}{2}\,h^2L_{i+1}^3(1-L_{i+1})^2\,u''_{i+1} \qquad (8.249)$$

The Galerkin equation for the unknown solution value u_{i+1} becomes

$$\int_{t_i}^{t_{i+1}}(10L_{i+1}^3-15L_{i+1}^4+6L_{i+1}^5)\left(\frac{du^{(e)}}{dt}-fu^{(e)}-g\right)dt=0 \qquad (8.250)$$

We substitute (8.249) into (8.250) and use (8.54) for the evaluation of the integrals and obtain

$$u_{i+1}+\frac{11}{42}h\,u'_{i+1}-\frac{1}{42}h^2u''_{i+1}$$
$$=u_i+\frac{11}{42}hu'_i+\frac{1}{42}\,h^2u''_i+hf^{(e)}\left[\frac{50}{231}u_i+\frac{302}{4620}h\,u'_i+\frac{362}{55440}h^2\,u''_i\right.$$
$$\left.+\frac{362}{462}u_{i+1}-\frac{622}{4620}h\,u'_{i+1}+\frac{562}{55440}\,h^2\,u''_{i+1}\right]+hg^{(e)} \qquad (8.251)$$

Stability analysis

We apply the difference schemes to the test equation

$$u'=\lambda u \qquad (8.252)$$

where λ is a constant.

For $\lambda < 0$, the growth factors (8.255) decrease for all values of $\bar{h} < 0$ and properly approximate the exact solution with accuracy depending on the order of the method. The methods are A-stable.

Next, we apply the scheme (8.246) to the test equation (8.252) and obtain

$$\left(1-\frac{8}{15}\bar{h}\right)u_{i+1}-\frac{4}{3}\left(1+\frac{1}{5}\bar{h}\right)u_i+\frac{1}{3}\left(1+\frac{2}{15}\bar{h}\right)u_{i-1}=0 \qquad (8.256)$$

The characteristic equation for (8.256) may be written as

$$\left(1-\frac{8}{15}\bar{h}\right)\xi^2-\frac{4}{3}\left(1+\frac{\bar{h}}{5}\right)\xi+\frac{1}{3}\left(1+\frac{2}{15}\bar{h}\right)=0 \qquad (8.257)$$

Substituting $\xi=(1+z)/(1-z)$ in the above equation we get

$$\left(\frac{8}{3}-\frac{2}{15}\bar{h}\right)z^2+\frac{4}{3}\left(1-\bar{h}\right)z-\frac{2}{3}\bar{h}=0 \qquad (8.258)$$

Using the Routh-Hurwitz criterion we find that the roots of the equation (8.258) for $\lambda < 0$, lie on the left half plane for all values of $\bar{h}$. Thus, the scheme (8.246) is A-Stable.

8.9.2 Second order initial value problems

We now consider the numerical methods for solving the linear second order initial value problem

$$\frac{d^2u}{dt^2}+f(t)\frac{du}{dt}+g(t)u=r(t), \qquad t\in[t_0, b]$$

$$u(t_0)=u_0, \qquad \frac{du(t_0)}{dt}=u_0' \qquad (8.259)$$

The Hermite cubic polynomial (8.247) is chosen as the approximate solution $u^{(e)}$. The solution values u_i and u_i' are known and we use the Galerkin method to determine the solution values u_{i+1} and u_{i+1}'. The Galerkin equations are given by

$$\int_{t_i}^{t_{i+1}} L_{i+1}^2(3-2L_{i+1})\left[\frac{d^2u^{(e)}}{dt^2}+f(t)\frac{du^{(e)}}{dt}+g(t)u^{(e)}-r\right]dt=0 \qquad (8.260)$$

$$\int_{t_i}^{t_{i+1}} (-hL_{i+1}^2L_i)\left[\frac{d^2u^{(e)}}{dt^2}+f(t)\frac{du^{(e)}}{dt}+g(t)u^{(e)}-r\right]dt=0 \qquad (8.261)$$

Assuming that the functions f, g and r are constants over the element (e) we have, after integration,

$$\begin{bmatrix} -504+210hf^{(e)} & 462h+42h^2f^{(e)} \\ +156h^2g^{(e)} & -22h^3g^{(e)} \\ -42+42hf^{(e)} & 56h-4g^{(e)}h^3 \\ +22h^2g^{(e)} & \end{bmatrix}\begin{bmatrix} u_{i+1} \\ \\ u_{i+1}' \end{bmatrix}$$

$$= \begin{bmatrix} -504+210h^2f^{(e)} - 54h^2g^{(e)} & -42h+42h^2f^{(e)} - 13h^3g^{(e)} \\ -42+42hf^{(e)} - 13h^2g^{(e)} & 14h+7h^2f^{(e)} - 3h^3g^{(e)} \end{bmatrix} \begin{bmatrix} u_i \\ u_i' \end{bmatrix} + \begin{bmatrix} 210 \\ 35 \end{bmatrix} h^2 r^{(e)} \tag{8.262}$$

We may discuss the stability of the method (8.262) by applying it to the test equation $u''+\lambda^2 u=0$ and adopting the procedure in Section 2.9.2. We obtain

$$\begin{bmatrix} -504+156\bar{h}^2 & 462h-22h\bar{h}^2 \\ -42+22\bar{h}^2 & 56h-4h\bar{h}^2 \end{bmatrix} \begin{bmatrix} u_{i+1} \\ u_{i+1}' \end{bmatrix} = \begin{bmatrix} -504-54\bar{h}^2 & -42h-13h\bar{h}^2 \\ -42-13\bar{h}^2 & 14h-3h\bar{h}^2 \end{bmatrix} \begin{bmatrix} u_i \\ u_i' \end{bmatrix} \tag{8.263}$$

which may be written as

$$\begin{bmatrix} u_{i+1} \\ u_{i+1}' \end{bmatrix} = \begin{bmatrix} a_{11} & ha_{12} \\ a_{21}/h & a_{22} \end{bmatrix} \begin{bmatrix} u_i \\ u_i' \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_i \\ u_i' \end{bmatrix}$$

where

$$\begin{aligned} a_{11} &= \frac{1}{|\mathbf{A}|}(8820-4074\bar{h}^2+70\bar{h}^4) \\ a_{12} &= \frac{1}{|\mathbf{A}|}(8820-1134\bar{h}^2+14\bar{h}^4) \\ a_{21} &= \frac{1}{|\mathbf{A}|}(-8820\bar{h}^2+840\bar{h}^4) \\ a_{22} &= \frac{1}{|\mathbf{A}|}(8820-4074\bar{h}^2+182\bar{h}^4) \\ |\mathbf{A}| &= 140(63+2.4\bar{h}^2+\bar{h}^4) \\ \bar{h}^2 &= h^2\lambda^2 \end{aligned} \tag{8.264}$$

Computing the eigenvalues of the matrix **A** as functions of $\bar{h}^2$, we find that the eignvalues are less than unit modulus for $0<\bar{h}^2<9.2$. Thus the stability interval of the method (8.262) when applied to the test equation $u''=-\lambda^2 u$ is, $0<\bar{h}^2<9.2$. The method does not have the interval of P-stability.

8.10 INITIAL BOUNDARY VALUE PROBLEMS

A simple method to solve the initial boundary value problems is to use the finite difference approximation in time with a finite element discretization in space. This method can be regarded as analysing conditions at a

In order to get the differential equation at the knot x_i, we write the element equations for the elements $x_{i-1} \leqslant x \leqslant x_i$ and $x_i \leqslant x \leqslant x_{i+1}$ and assemble these element equations. We obtain

$$\frac{1}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_{i-1} \\ \dot{u}_i \\ \dot{u}_{i+1} \end{bmatrix}+\frac{1}{h^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_{i-1} \\ u_i \\ u_{i+1} \end{bmatrix}=0$$

We put the row corresponding to the knot i to zero and write the semi-discretization as

$$\frac{1}{6}(\dot{u}_{i-1}+4\dot{u}_i+\dot{u}_{i+1})=\frac{1}{h^2}(u_{i-1}-2u_i+u_{i+1})$$

Applying the trapezoidal rule we obtain the Crank-Nicolson method

$$\frac{1}{6}\left[\left(u_{i-1}^{n+1}-u_{i-1}^{n}\right)+4(u_i^{n+1}-u_i^n)+(u_{i+1}^{n+1}-u_{i+1}^{n})\right.$$

$$\left.=\frac{r}{2}\left[(u_{i-1}^{n+1}+u_{i-1}^{n})-2(u_i^{n+1}+u_i^n)+(u_{i+1}^{n+1}+u_{i+1}^{n})\right]\right.$$

or

$$(1-3r)u_{i-1}^{n+1}+(4+6r)u_i^{n+1}+(1-3r)u_{i+1}^{n+1}$$

$$=(1+3r)u_{i-1}^{n}+(4-6r)u_i^n+(1+3r)u_{i+1}^{n}$$

For $h=\frac{1}{2}$, the nodal points (x_i, t_n)

are shown in Figure 8.19, where $i=1(1)5$, $n=0, 1, 2, \cdots$

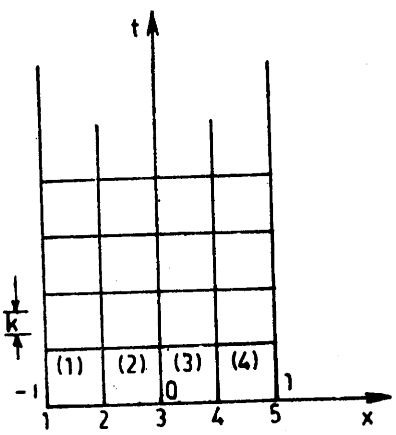

Fig. 8.19 Representation of nodal points

For $r=\frac{1}{2}$, we have

$$-\frac{1}{2}u_{i-1}^{n+1}+7u_i^{n+1}-\frac{1}{2}u_{i+1}^{n+1}=\frac{5}{2}u_{i-1}^n+u_i^n+\frac{5}{2}u_{i+1}^n$$

$$i=3,\ n=0,-\frac{1}{2}u_2^1+7u_3^1-\frac{1}{2}u_4^1=\frac{5}{2}u_2^0+u_3^0+\frac{5}{2}u_4^0$$

$$i=4,\ n=0,-\frac{1}{2}u_3^1+7u_4^1-\frac{1}{2}u_5^1=\frac{5}{2}u_3^0+u_4^0+\frac{5}{2}u_5^0$$

incorporating the initial and boundary conditions and the symmetric conditions, we get

$$7u_3^1-u_4^1=u_3^0+5u_4^0$$

$$-\frac{1}{2}u_3^1+7u_4^1=\frac{5}{2}u_3^0+u_4^0$$

or

$$\begin{bmatrix} 7 & -1 \\ -\frac{1}{2} & 7 \end{bmatrix}\begin{bmatrix} u_3^1 \\ u_4^1 \end{bmatrix}=\begin{bmatrix} 1 & 5 \\ \frac{5}{2} & 1 \end{bmatrix}\begin{bmatrix} u_3^0 \\ u_4^0 \end{bmatrix}$$

Simplifying, we obtain

$$\begin{bmatrix} u_3^1 \\ u_4^1 \end{bmatrix}=\frac{1}{97}\begin{bmatrix} 19 & 72 \\ 36 & 19 \end{bmatrix}\begin{bmatrix} u_3^0 \\ u_4^0 \end{bmatrix}$$

or

$$\begin{bmatrix} u_3^1 \\ u_4^1 \end{bmatrix}=\begin{bmatrix} 0.7207 \\ 0.5096 \end{bmatrix}$$

For, $n=1$, we have

$$\begin{bmatrix} u_3^2 \\ u_4^2 \end{bmatrix}=\frac{1}{97}\begin{bmatrix} 19 & 72 \\ 36 & 19 \end{bmatrix}\begin{bmatrix} u_3^1 \\ u_4^1 \end{bmatrix}=\begin{bmatrix} 0.5194 \\ 0.3673 \end{bmatrix}$$

8.10.2 First order hyperbolic equation

We consider the first order hyperbolic partial differential equation of the form

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \tag{8.272}$$

with appropriate initial and boundary condition, where c is a constant. We use the linear piecewise approximation in the space variable and the Galerkin method to obtain the semi-discrete approximation to (8.272). We have

$$u^{(e)}=N_j(x)u_j(t)+N_k(x)u_k(t)$$

where

$$N_j = \frac{x_k - x}{l^{(e)}}, \qquad N_k = \frac{x - x_j}{l^{(e)}},$$

$$l^{(e)} = x_k - x_j$$

We also have

$$\frac{\partial u}{\partial t} = N_j \frac{du_j}{dt} + N_k \frac{du_k}{dt}$$

The Galerkin equations in matrix form may be written as

$$\int_{x_j}^{x_k} \left\{ \begin{bmatrix} N_j N_j & N_j N_k \\ N_k N_j & N_k N_k \end{bmatrix} \begin{bmatrix} \dot{u}_j \\ \dot{u}_k \end{bmatrix} + c \begin{bmatrix} N_j N_j' & N_j N_k' \\ N_k N_j' & N_k N_k' \end{bmatrix} \begin{bmatrix} u_j \\ u_k \end{bmatrix} \right\} dx = \mathbf{0} \tag{8.273}$$

We use the relations (8.54) to simplify (8.273) and obtain the element equations

$$\frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_j \\ \dot{u}_k \end{bmatrix} + \frac{c}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_j \\ u_k \end{bmatrix} = \mathbf{0} \tag{8.274}$$

We assemble the element equations for the elements

$$x_{i-1} \leqslant x \leqslant x_i \text{ and } x_i \leqslant x \leqslant x_{i+1}$$

and obtain

$$\frac{1}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_{i-1} \\ \dot{u}_i \\ \dot{u}_{i+1} \end{bmatrix} + \frac{c}{2h} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \\ u_{i+1} \end{bmatrix} = \mathbf{0} \tag{8.275}$$

The finite element difference-differential equation at the node 'i' for (8.272) becomes

$$\frac{1}{6} (\dot{u}_{i-1} + 4\dot{u}_i + \dot{u}_{i+1}) + \frac{c}{2h} (-u_{i-1} + u_{i+1}) = 0$$

or

$$\left(1 + \frac{1}{6}\delta_x^2\right) \dot{u}_i + \frac{c}{h} \mu_x \delta_x u_i = 0 \tag{8.276}$$

The Crank-Nicolson discretization may be written as

$$\left[1 + \frac{1}{6}\delta_x^2 + \frac{1}{2} cp\mu_x\delta_x\right] u_i^{n+1} = \left[1 + \frac{1}{6}\delta_x^2 - \frac{cp}{2}\mu_x\delta_x\right] u_i^n \tag{8.277}$$

where $p = \dfrac{k}{h}$.

Applying the von Neumann method to (8.277), the characteristic equation is given by

$$\left(1-\frac{2}{3}\sin^2\frac{\theta h}{2}+\frac{1}{2}cpi\sin\theta h\right)\xi=\left(1-\frac{2}{3}\sin^2\frac{\theta h}{2}-\frac{1}{2}cpi\sin\theta h\right) \quad (8.278)$$

We obtain $|\xi| = 1$

The difference scheme (8.277) is unconditionally stable and nondissipative.

Next we take the piecewise approximate solution over the typical element (e) with nodes ijk (see Fig. 8.12(a)) in the form (8.202).

$$u^{(e)} = N_i u_i + N_j u_j + N_k u_k$$

The Galerkin element equation (8.272) may be written as

$$\left(\frac{1}{6}\begin{bmatrix} c_i & c_j & c_k \\ c_i & c_j & c_k \\ c_i & c_j & c_k \end{bmatrix}+\frac{c}{6}\begin{bmatrix} b_i & b_j & b_k \\ b_i & b_j & b_k \\ b_i & b_j & b_k \end{bmatrix}\right)\begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix}=0 \quad (8.279)$$

We now obtain the difference equation for (8.272) at the node '2' for the triangular network as shown in Figure 8.20. We assume the steplengths h and k in the x and t directions respectively. The necessary information required for developing the difference equation for (8.272) at '2' for the triangular network as shown in Figure 8.20(a) are listed in Table 8.12.

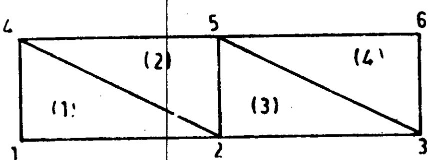

Fig. 8.20.(a) Triangular network

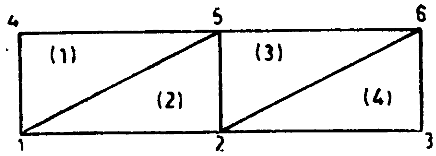

Fig. 8.20(b) Triangular network

TABLE 8.12 NODAL PARAMETERS FOR ELEMENT DIFFERENCE EQUATION

Element (e)	Nodes i	j	k
(1)	1	2	4
(2)	4	2	5
(3)	2	3	5
(4)	5	3	6

(e)	Coordinates x_i	x_j	x_k	t_i	t_j	t_k
(1)	$-h$	0	$-h$	0	0	k
(2)	$-h$	0	0	k	0	k
(3)	0	h	0	0	0	k
(4)	0	h	h	k	0	k

(e)	Parameters b_i	b_j	b_k	c_i	c_j	c_k
(1)	$-k$	k	0	$-h$	0	h
(2)	$-k$	0	k	0	$-h$	h
(3)	$-k$	k	0	$-h$	0	h
(4)	$-k$	0	k	0	$-h$	h

On simplification the linear difference scheme is given by

$$\left[1-\frac{p}{2}(u_m^n+2u_{m-1}^n)\right]v_{m-1}^n+\left[4+\frac{p}{2}(u_{m+1}^n-u_{m-1}^n)\right]v_m^n$$

$$+\left[1+\frac{p}{2}(2u_{m+1}^n+u_m^n)\right]v_{m+1}^n$$

$$=-p(u_{m+1}^n-u_{m-1}^n)(u_{m+1}^n+u_m^n+u_{m-1}^n),\quad m=1(1)M$$

We also have

$$u_m^{n+1}=u_m^n+k\left(\frac{\partial u}{\partial t}\right)_m^n+\frac{1}{2}k^2\left(\frac{\partial^2 u}{\partial t^2}\right)_m^n+0(k^3)$$

$$=u_m^n+k\left(-\frac{1}{2}\frac{\partial u^2}{\partial x}\right)_m^n+\frac{1}{2}k^2\left(-\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}-u\frac{\partial^2 u}{\partial x\partial t}\right)$$

$$=u_m^n-\frac{1}{2}p\left(h\frac{\partial u^2}{\partial x}\right)_m^n+\frac{1}{2}p^2\left(\frac{1}{2}h^2\frac{\partial u^2}{\partial x}\frac{\partial u}{\partial x}+u^2h^2\frac{\partial^2 u}{\partial x^2}\right.$$

$$\left.+u\left(h\frac{\partial u}{\partial x}\right)^2\right)+0(k^3)$$

The numerical boundary condition at $x=1$ may be written as

$$v_{M+1}^n=-\frac{1}{2}p((u^2)_{M+1}^n-(u^2)_M^n)+\frac{1}{2}p^2\left(\frac{1}{2}((u^2)_{M+1}^n-(u^2)_M^n)(u_{M+1}^n-u_M^n)\right.$$

$$\left.+(u^2)_{M+1}^n(u_{M+1}^n-2u_M^n+u_{M-1}^n)+u_{M+1}^n(u_{M+1}^n-u_M^n)^2\right)$$

For $h=\frac{1}{4}$ and $p=\frac{1}{2}$, the initial conditions become

$$u_0^0=0,\quad u_1^0=\frac{1}{16},$$

$$u_2^0=\frac{1}{4},\ u_3^0=\frac{9}{16},\ u_4^0=1$$

The boundary conditions give

$$v_0^0=u_0^1-u_0^0=0$$

$$v_4^0=u_4^1-u_4^0$$

$$=-\frac{1}{4}\left(1-\frac{81}{256}\right)+\frac{1}{8}\left[\frac{1}{2}\left(1-\frac{81}{256}\right)\left(1-\frac{9}{16}\right)\right.$$

$$\left.+\left(1-2\frac{9}{16}+\frac{1}{4}\right)+\left(1-\frac{9}{16}\right)^2\right]$$

$$=-0.1127$$

$$u_4^1=u_4^0-0.1127=0.8873$$

For $n=0$, $p=\frac{1}{2}$, we have

$$m=1,\ \left[1-\frac{1}{4}(u_1^0+2u_0^0)\right]v_0^0+\left[4+\frac{1}{4}(u_2^0-u_0^0)\right]v_1^0+\left[1+\frac{1}{4}(2u_2^0+u_1^0)\right]v_2^0$$
$$=-\frac{1}{2}(u_2^0-u_0^0)(u_2^0+u_1^0+u_0^0)$$
$$m=2,\ \left[1-\frac{1}{4}(u_2^0+2u_1^0)\right]v_1^0+\left[4+\frac{1}{4}(u_3^0-u_1^0)\right]v_2^0+\left[1+\frac{1}{4}(2u_3^0+u_2^0)\right]v_3^0$$
$$=-\frac{1}{2}(u_3^0-u_1^0)(u_3^0+u_2^0+u_1^0)$$
$$m=3,\ \left[1-\frac{1}{4}(u_3^0+2u_2^0)\right]v_2^0+\left[4+\frac{1}{4}(u_4^0-u_2^0)\right]v_3^0+\left[1+\frac{1}{4}(2u_4^0+u_3^0)\right]v_4^0$$
$$=-\frac{1}{2}(u_4^0-u_2^0)(u_4^0+u_3^0+u_2^0)$$

Simplifying, we get the system of equations

$$520v_1^0+146v_2^0=-5$$
$$29v_1^0+132v_2^0+43v_3^0=-7$$
$$47v_2^0+268v_3^0=-31.6712$$

Solving, we obtain

$$v_1^0=-0.0057,\quad v_2^0=-0.0141,\quad v_3^0=-0.1157$$

Thus we get

$$u_1^1=0.0568,\quad u_2^1=0.2359,\quad u_3^1=0.4468$$

The analytic solution $u(x, t)=(1+2xt-(1+4xt)^{1/2})/2t^2$ gives

$$u\left(\frac{1}{4},\frac{1}{8}\right)=0.0589,\quad u\left(\frac{1}{2},\frac{1}{8}\right)=0.2229$$
$$u\left(\frac{3}{4},\frac{1}{8}\right)=0.4767$$

8.10.3 Second order hyperbolic equation

We consider the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} \tag{8.284}$$

with appropriate initial and boundary conditions.

The variational formulation is given by

$$J[u]=\frac{1}{2}\iint_{\mathfrak{R}}\left(\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial t}\right)^2\right)dx\,dt=\text{minimum} \tag{8.285}$$

We use the triangular elements in x-t plane. The element equations obtained in Section 8.7 may be used here. For example for a typical triangular by the following methods.

(i) Least square method
(ii) Galerkin method
(iii) Collocation method by using $x = 1/4$ and $x = 3/4$ as collocation points.

(b) Write the approximate solution function in the form

$$w(x) = N_1(x)u_1 + N_2(x)u_2$$

where u_1 and u_2 denote the values of u at

$$x = 1/3 \text{ and } 2/3.$$

2. Consider the boundary value problem

$$u'' = u - 4xe^x$$
$$u'(0) - u(0) = 1$$
$$u'(1) + u(1) = -e$$

Write the approximate solution in the form

$$w(x) = N_0(x) + N_1(x)u(0) + N_2(x)u(1)$$

that satisfies the boundary conditions and determine it with the help of the Galerkin method.

3. Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 100x$$

$$u(x, 0) = 0, \qquad 0 \leqslant x \leqslant 1$$
$$u(0, t) = u(1, t) = 0, \qquad t > 0$$

with the approximate solution in the form

$$w(x, t) = N_1(x)u_1(t) + N_2(x)u_2(t)$$

where $u_1(t)$ and $u_2(t)$ are the unknown solution values at the nodes 1/4 and 3/4 respectively.

(a) Determine the interpolating functions $N_1(x)$ and $N_2(x)$.
(b) Use the Galerkin method to get the system of first order differential equations

$$\dot{\boldsymbol{\phi}} = \mathbf{A}\boldsymbol{\phi} + \mathbf{b}$$

where $\boldsymbol{\phi} = [u_1\ u_2]^T$, $\mathbf{A}$ is a real 2×2 matrix and $\mathbf{b}$ is 2×1 matrix. Find the matrices $\mathbf{A}$ and $\mathbf{b}$.

4. Apply the Galerkin method to the boundary value problem

$$\nabla^2 u + \lambda u = 0, \quad |x| \leqslant 1, \quad |y| \leqslant 1$$
$$u = 0, \quad |x| = 1, \quad |y| = 1$$

to get the characteristic equation in the form

$$|\mathbf{A} - \lambda \mathbf{B}| = 0$$

(a) The approximate solution is written as

$$w(x, y) = N_1u_1 + N_2u_2 + N_3u_3$$

where u_1, u_2 and u_3 denote the values of u at the nodes (0, 0), (1/2, 0) and (1/2, 1/2) respectively. Using the symmetry of the problem, determine the interpolating functions N_1, N_2 and N_3.

(b) Obtain the matrices **A** and **B**.

5. (a) Find the variational functional for the boundary value problem

$$u'' = u - 4xe^x$$
$$u'(0) - u(0) = 1, \qquad u'(1) + u(1) = -e$$

(b) Write the approximate solution in terms of two unknown function values of u that does not satisfy the above boundary conditions.

(c) Determine the approximate solution with the help of the Ritz method.

6. Find the variational functional for the following boundary value problem

$$y'' = \frac{3}{2} y^2$$
$$y(0) = 4, \ y(1) = 1$$

Determine the approximate solution of the form

$$w(x) = (2x-1)(5x-4) + 4x(1-x)u_2$$

where u_2 is the unknown function value at the node $\frac{1}{2}$.

7. The application of the finite element method to the boundary value problem

$$-u'' = x$$
$$u(0) = u(1) = 0$$

leads to the system of equations

$$\mathbf{Au} = \mathbf{b}$$

Determine the matrix **A** and the column vector **b** for two, four and six elements of equal lengths, using the linear line segment element.

8. Consider the boundary value problem

$$-u'' + u = x$$
$$u'(0) = 1, \qquad u'(1) = 2$$

Apply the Galerkin method to compute the finite element approximation for two and four elements. Use the linear shape functions.

9. Set up the equations for the finite element solution of the boundary value problem

$$-\frac{d^2u}{dx^2} + fu + g = 0,$$
$$u(0) = u(1) = 0$$

14. The piecewise interpolating polynomial function $u^{(e)}(x)$ which satisfies the following relations;
   (i) the differential equation,

$$u''^{(e)} + \alpha u^{(e)} = \frac{(x - x_{j-1})}{h}(M_j + \alpha u_j) + \frac{(x_j - x)}{h}(M_{j-1} + \alpha u_{j-1}), \ \alpha > 0$$

   (ii) the interpolating conditions

$$u^{(e)}(x_{j-1}) = u_{j-1}, \qquad u^{(e)}(x_j) = u_j$$

   (iii) the continuity condition

$$u'^{(e)}(x_j+) = u'^{(e)}(x_j-)$$

   (iv) $$\frac{w}{2} = \tan\frac{w}{2}, \ w = \sqrt{\alpha}\, h$$

   where $u''^{(e)}(x_j) \doteq M_j$, (e) denotes a line segment element, $x_{j-1} \leqslant x \leqslant x_j$ and h is the length of element
   is called the *spline in compression*.
   (a) Prove that the continuity of the first derivative of the function $u^{(e)}(x)$ at $x = x_j$ gives

$$u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{4}(M_{j-1} + 2M_j + M_{j+1})$$

   (b) Show that the other spline relations may be written as

   (i) $m_{j+1} - m_j = \dfrac{h}{2}(M_{j+1} + M_j)$

   (ii) $u_{j+1} - u_j = \dfrac{h}{2}(m_{j+1} + m_j)$

   where $\quad m_j = u'^{(e)}(x_j)$.

15. (a) Obtain the piecewise quintic Hermite interpolation polynomial by matching the function values as well as the first two derivative values at each of the two nodes of the line segment element (e), $x_j \leqslant x \leqslant x_k$.
   (b) Derive the boundary value problem that characterizes the minima of the functional

$$J[u] = \frac{1}{2}\int_0^1 [(u'')^2 - 2(u')^2 + u^2 - 2u]\, dx$$

$$u(0) = u'(0) = 0, \qquad u(1) = u'(1) = 0$$

   (c) Develop the linear system of equations describing an approximation of the problem using only two finite elements and piecewise Hermite quintic approximate function.

16. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} + \lambda u - Q = 0$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0$$

The coefficients α, λ and Q are known parameters. Assuming the $\bar{u}$ varies linearly in the element (e), use the Galerkin method with linear shape functions to obtain the element equation in the form

$$\mathbf{A}^{(e)}\begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix} + \mathbf{S}^{(e)}\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \mathbf{b}^{(e)}$$

where x_i and x_j are the coordinates of the end nodes of the element (e) and a dot denotes differentiation with respect to t.
Find the matrices $\mathbf{A}^{(e)}$, $\mathbf{S}^{(e)}$ and $\mathbf{b}^{(e)}$.

17. Compute the finite element solution of the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -1 < x < 1, \quad t > 0$$

$$u(-1, t) = u(1, t) = 0$$

$$u(x, 0) = 100\cos\frac{\pi x}{2}$$

by using six elements in the x-direction with linear shape functions. Apply the finite differencing to the time derivative to get the difference equations. Choose $\Delta t = \frac{1}{18}$ and integrate until $t = \frac{1}{9}$.

18. Consider the heat flow equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + cu$$

subject to the initial and boundary conditions

$$u(x, 0) = f(x) \qquad 0 \leqslant x \leqslant 1$$

$$u(0, t) = u(1, t) = 0, \quad t > 0$$

Divide [0, 1] into $N+1$ elements with element length h. Treating $\partial u/\partial t$ as a constant and using linear piecewise polynomial, set up the finite element equation

$$\mathbf{B}\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$$

where $\mathbf{B}$ and $\mathbf{A}$ are $N \times N$ matrices and $\mathbf{u} = [u_1\ u_2 \cdots u_N]^T$.

(a) Find the matrices **A** and **B**

(b) Use finite differencing in time to get the finite difference matrix equation and discuss its stability.

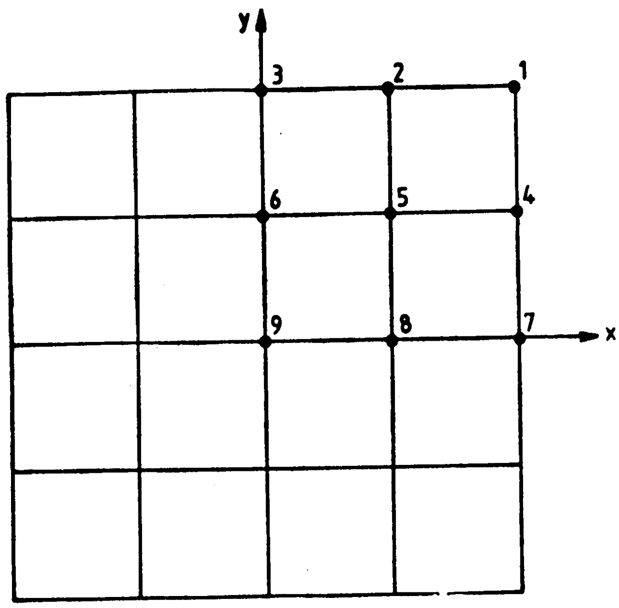

Fig. 8.23 Square elements

25. Consider the boundary value problem

$$\nabla^2 u = -1$$

subject to the condition

$$u = 0$$

on the boundary described by the lines $y=0$, $y=2-\sqrt{3}\,x$, $y=\sqrt{3}\,x$. Obtain the finite element equations for the following configuration of triangular elements. (see Fig. 8.24). The node 4 is at the centroid of the equilateral triangle 1 2 3.

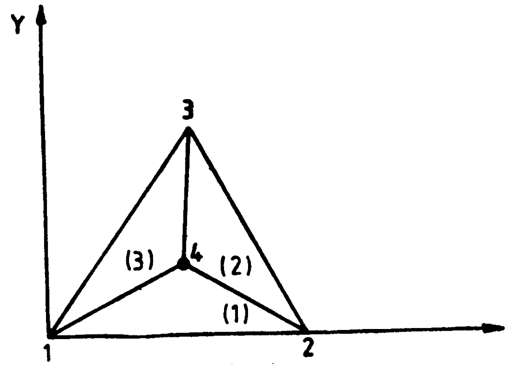

Fig. 8.24 Triangular network

26. Use the finite element method to obtain the difference scheme at the node '0' of the differential equation

$$\nabla^2 u + k^2 u = 0$$

for the following configuration (square) of the triangular elements (see Fig. 8.25).

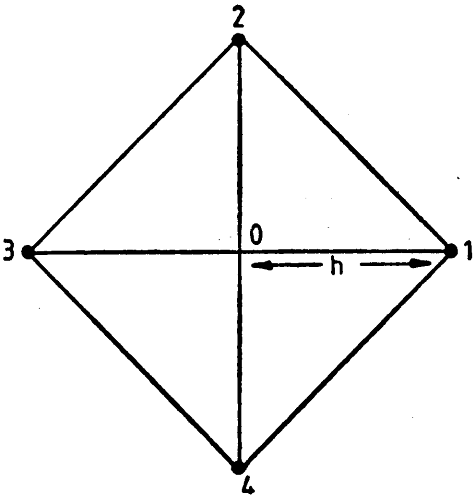

Fig. 8.25 Triangular network

27. (a) Determine the piecewise quadratic approximating function of the form

$$u^{(e)} = \sum_{i=1}^{8} N_i u_i$$

for the following configurations of rectangular finite elements (see Fig. 8.26)

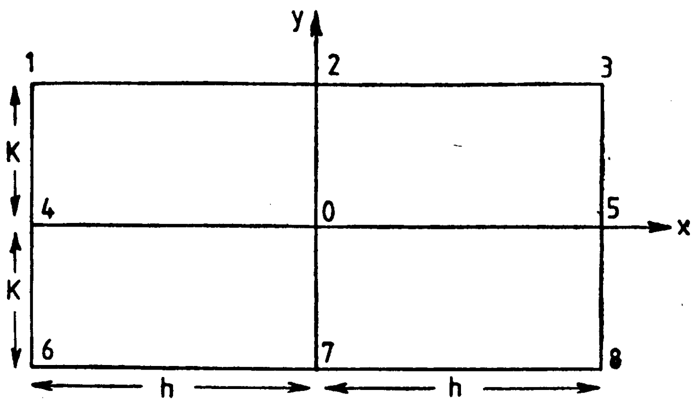

Fig. 8.26 Rectangular network

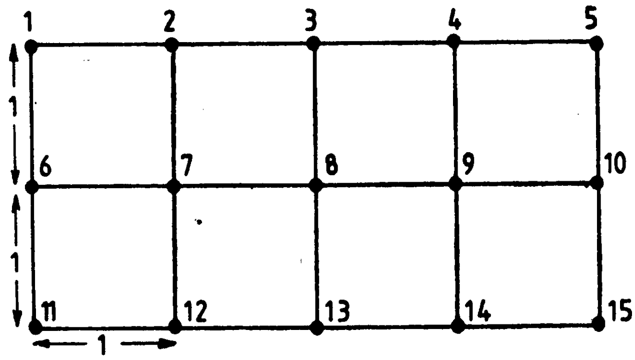

Fig. 8.29 Square network

31. Determine the necessary conditions to be satisfied by the solution of the variational problem

$$J=\int_a^b \left(pu'^2+qu^2-2\gamma\int_0^u f(t)\,dt\right)dx+\frac{p(a)}{\alpha_1}(-\alpha_0u^2(a)+2\gamma_1u(a))$$

$$+\frac{p(b)}{\beta_1}(\beta_0u^2(b)-2\gamma_2u(b))$$

where $u=u(x)$; p, q and γ are prescribed functions of x; f is a prescribed function of t; α_0, α_1, β_0, β_1, γ_1 and γ_2 are constants; and a prime denotes differentiation with respect to x.

32. Obtain the element equations for the linear line segment element (see Fig. 8.9) for the boundary value problem

$$\frac{d^2u}{dx^2}-\frac{du}{dx}-u^2=0$$

$$u(0)=1, \qquad u'(1)=0$$

Use Galerkin's method and, (i) the nonlinear differential equation; (ii) the quasilinear differential equation.

33. Using the Galerkin method and the cubic Hermite polynomial, derive the element equations for the boundary value problem

$$u'''-(u')^2+1=0$$

$$u(0)=u'(0)=0$$

$$u'(\infty)=1$$

Consider the quasilinear differential equation.

34. Discuss the stability of the difference equations of the boundary value problem

$$u'''+Ku''=0$$

$$u(0)=u'(0)=0, \qquad u'(\infty)=1$$

where $K > 0$ is a constant. Use the Galerkin method with the cubic Hermite polynomial.

35. Derive the Euler equation and the boundary conditions to be satisfied by the solution of the variational problem

$$J=\int_a^b\left(p(x)(y'')^2+q(x)(y')^2+2r(x)\int_0^y f(t)\,dt\right)dx$$

where $p(a) \neq 0$, $p(b) \neq 0$.

36. Obtain the difference schemes for the boundary value problem

$$y^{iv}-Ky''=x$$
$$y(0)=y'(0)=0, \qquad y(1)=y'(1)=0$$

where K is a constant. Also discuss the stability of the difference schemes. Use the variational finite element method with the cubic Hermite polynomial.

37. Find the element equations for the boundary value problem

$$u^{iv}=\frac{3}{8}u^5$$
$$u(0)=4, \qquad u(1)=2$$
$$u'(0)=-4 \qquad u'(1)=-1$$

using the cubic Hermite polynomial and the quasilinear differential equation.

38. Obtain the difference approximation for the Laplace equation $\nabla^2 u=0$ at the node '0' for the two networks as shown in Fig. 8.30.

39. The Laplace equation $\nabla^2 u=0$ over a regular hexagonal network using the difference method may be replaced by

$$\frac{2}{3h^2}\left(\sum_{i=1}^{6} u_i-6u_0\right)=0$$

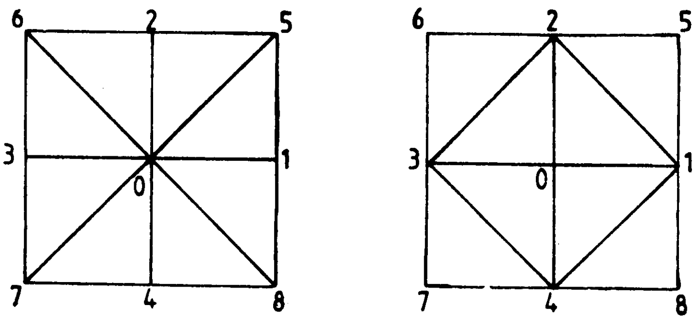

Fig. 8.30 Triangular network

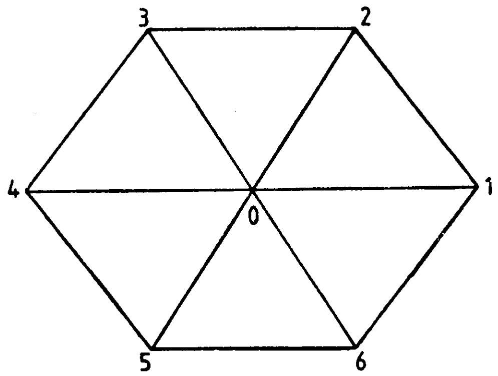

Fig. 8.31 Hexagonal network

Prove that the same relationship may be obtained by using three node equilateral triangular finite elements (see Fig. 8.31).

40. Obtain the necessary conditions to be satisfied by the solution of the variational problem

$$J=\iint_{\mathcal{R}}\left\{p(x, y)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)+q(x, y)u^2-2r(x, y)\int_0^u f(t)\, dt\right\} dx\, dy$$

$$+\int_{\partial\mathcal{R}}\frac{p(s)}{a_3(s)}(-a_1(s)u^2+2a_4(s)u)\, ds$$

where $\partial\mathcal{R}$ is the closed boundary of the region $\mathcal{R}$ and s is the arc length of $\partial\mathcal{R}$ measured from a fixed point.

41. Find the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1+e^u}{2}=0, \quad |x|\leqslant 1, \quad |y|\leqslant 1$$

$$u=0, \quad |x|=1, \quad |y|=1$$

Use the three node triangular element and, (i) the nonlinear differential equation; (ii) the quasilinear differential equation; with $h=1$.

42. Take the approximate function $u^{(e)}$ on the triangular element with one curved side as shown in Figure 8.12(d) as

$$u^{(e)}=L_1(1-2L_3)u_i+L_2u_j+L_3(1-2L_1)u_k+4L_1L_3u_l$$

where L_1, L_2 and L_3 are the local coordinates. Deduce the element equations for the curved triangular elements (see Fig. 8.16(b)) arising in solving the boundary value problem

$$\nabla^2 u=\frac{3}{2}u^2 \quad \text{in } \mathcal{R}$$

$$u=1 \quad \text{on } \partial\mathcal{R}$$

where the domain $\mathscr{R}$ is defined by

$$x^2+y^2 \leqslant 1, \qquad x \geqslant 0, \qquad y \geqslant 0$$

with $h=\frac{1}{2}$. Use the quasilinear differential equation.

43. Find the element equations corresponding to a three node axisymmetric triangular element for the Laplace equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

44. Derive the element equations corresponding to a four node tetrahedron element for the Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+K^2u=0$$

where K is a constant.

45. Form the variational problem of the differential equation

$$(1+u_y^2)u_{xx}-2u_xu_yu_{xy}+(1+u_x^2)u_{yy}=0$$

Use the three node triangular element as shown in Figure 8.12(a) to find the element equations.

46. Determine the Euler equation and the natural boundary conditions for the functional

$$J=\frac{1}{2}\int_{t_1}^{t_2}\int_0^l\left[\left(\frac{\partial u}{\partial t}\right)^2-\left(\frac{\partial u}{\partial x}\right)^2\right]dx\,dt$$

Also obtain the element equations corresponding to the three node triangular element. Discuss the stability of the difference scheme for node '0' as shown in Figure 8.13(a)

47. Consider the variational problem

$$J=\frac{1}{2}\int_{t_1}^{t_2}\int_0^l\left[\left(\frac{\partial u}{\partial t}\right)^2-\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right]dx\,dt$$

Find the necessary conditions to be satisfied by the solution of the variational problem. Also determine the element equations using the cubic Hermite polynomial (8.98) defined over the four node rectangular element. Discuss the stability of the difference scheme for the node '0' as shown in Figure 8.13(b).

48. Obtain the Euler equation and the natural boundary conditions for the functional

$$J=\int_a^b\int_c^d\left[\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)^2+\alpha\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}+(1-\alpha)\left(\frac{\partial^2 u}{\partial x\,dy}\right)^2\right]dx\,dy=0$$

where α is a constant.

49. Derive the element equations for the differential equation

$$\nabla^4 u = 1$$

corresponding to the rectangular element (e) as shown in Figure 8.32,

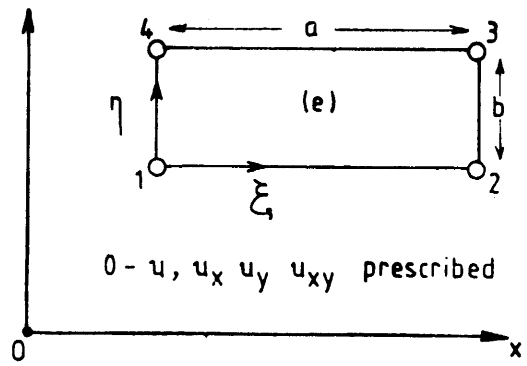

Fig. 8.32 Rectangular element

with the piecewise approximate function

$$u^{(e)} = N_1 u_1 + N_2 u_{1,x} + N_3 u_{1,y} + N_4 u_{1,xy} + \cdots + N_{16} u_{4,xy}$$

where

$$\begin{aligned} N_1 &= f_1(\xi)f_1(\eta) & N_2 &= ag_1(\xi)f_1(\eta) \\ N_3 &= bf_1(\xi)g_1(\eta) & N_4 &= abg_1(\xi)g_1(\eta) \\ N_5 &= f_2(\xi)f_1(\eta) & N_6 &= ag_2(\xi)f_1(\eta) \\ N_7 &= bf_2(\xi)g_1(\eta) & N_8 &= abg_2(\xi)g_1(\eta) \\ N_9 &= f_2(\xi)f_2(\eta) & N_{10} &= ag_2(\xi)g_2(\eta) \\ N_{11} &= bf_2(\xi)g_2(\eta) & N_{12} &= abg_1(\xi)f_2(\eta) \\ N_{13} &= f_1(\xi)f_2(\eta) & N_{14} &= ag_1(\xi)f_2(\eta) \\ N_{15} &= bf_1(\xi)g_2(\eta) & N_{16} &= abg_1(\xi)g_2(\eta) \end{aligned}$$

and

$$f_1(t) = 1 - 3t^2 + 2t^3 \qquad g_1(t) = t - 2t^2 + t^3$$

$$f_2(t) = 3t^2 - 2t^3 \qquad g_2(t) = t^3 - t^2$$

50. Obtain the differential equation corresponding to the functional

$$J = \int_{t_1}^{t_2} \int_0^l \int_0^l \left(\left(\frac{\partial u}{\partial t} \right)^2 - \left(\nabla^2 u \right)^2 \right) dx\,dy\,dt$$

Determine the essential conditions to be satisfied by the solution of the variational problem.